Complements in systems with radix $R$

- For a non-negative $(N+M)$-positional number
  
  \[
  (A)_R = (a_{N-1}a_{N-2} \ldots a_1a_0.a_{-1} \ldots a_{-M})_R
  \]

  there are defined two types of complements:

  - $R$’s complement
    
    \[
    \overline{A} = R^N - A \text{ for } A \neq 0 \text{ (for } A = 0, \overline{A} = 0) \]
    
    \[
    \overline{A} = A + R^{-M}
    \]

  - (R-1)’s complement (diminished complement)
    
    \[
    \overline{A} = R^N - A - R^{-M}
    \]
Representation of numbers with sign

- Sign digit and numeric digits
- Notations
  - Sign-Magnitude (SM)
  - Diminished Radix Complement (DRC)
  - Radix Complement (RC)
- \((N+M)\)-positional number

\[ X_R = \pm(x_{N-1}x_{N-2}\ldots x_1x_0.x_{-1}\ldots x_{-M})_R = \pm \sum_{i=-M}^{N-1} x_i \cdot R^i = \pm |X| \]

in system with radix \(R\) is represented as

\[ X^{SM,DRC,RC} = x_N x_{N-1}^{'} x_{N-2}^{'} \ldots x_1^{'} x_0^{'} . x_{-1}^{'} \ldots x_{-M}^{'} = x_N \cdot R^N + |X'| \]

where \(x_N\) is the sign digit, values of digits \(x_i^{'}\) depend on the used notation, and

\[ |X'| = \sum_{i=-M}^{N-1} x_i^{'} \cdot R^i \]
Sign-Magnitude notation

- Two numbers of the same magnitude but with opposite signs differ only at the sign digit position
  \[ X^{SM} = x_N \cdot R^N + |X| \]
- Zero can be represented in two ways:
  - positive zero \( +0 = 000\ldots0.0\ldots0 \)
  - negative zero \( -0 = (R - 1)00\ldots0.0\ldots0 \)
Diminished Radix Complement notation

• $X$ is represented as

$$X^{DRC} = \begin{cases} 
0 \cdot R^N + |X| & \text{for positive } X \\
(R - 1) \cdot R^N + |X| & \text{for negative } X 
\end{cases}$$

• Zero can be represented in two ways:
  – positive zero
    $$+0 = 000\ldots0.0\ldots0$$
  – negative zero
    $$-0 = (R - 1)(R - 1)(R - 1)\ldots(R - 1).(R - 1)\ldots(R - 1)$$
Radix Complement notation

• $X$ is represented as

$$X_{RC} = \begin{cases} 0 \cdot \overline{R^N} + |X| & \text{for positive } X \\ (R - 1) \cdot R^N + |\overline{X}| & \text{for negative } X \end{cases}$$

• One way to represent zero

$$+0 = 000\ldots0.0\ldots0$$
Characteristics of notations

• For positive $X$ representation is the same

$$X^{SM} = X^{DRC} = X^{RC} \quad \text{and} \quad |X'| = |X| = \sum_{i=-M}^{N-1} x_i \cdot R^i$$

• For negative $X$

$$|X'| = \begin{cases} 
|\bar{X}| = \sum_{i=-M}^{N-1} \bar{x_i} \cdot R^i & \text{for DRC notation} \\
|\bar{X}| = \sum_{i=-M}^{N-1} \bar{x_i} \cdot R^i + R^{-M} & \text{for RC notation}
\end{cases}$$

and

$$X^{RC} = X^{DRC} + R^{-M}$$
Numbers in digital systems

• Fixed-point notation – the placement of the radix point within the word representing a number is constant, *fixed*
  – advantage – simplicity
  – disadvantage – problems with representation of very large and very small numbers, loss of precision

• Floating point notation – the placement of the radix point within the word representing a number can change – it *floats*
  – advantage – can easily be used to store very large and very small numbers
  – disadvantage – multiple representations for a number

\[ \text{sign} \cdot \text{mantissa} \cdot \text{radix}^{\text{exponent}} \]
Fixed-point format of a number

a) \((N + M + 1)\)-positional mixed number

\[
\begin{array}{cccccc}
\text{Sign field} & \text{Numeric field} \\
& x_S & x_{N-1} & \cdots & x_0 & x_{-1} & \cdots & x_{-M} \\
\end{array}
\]

b) \((N + 1)\)-positional integer number

\[
\begin{array}{cccc}
\text{Sign field} & \text{Numeric field} \\
& x_S & x_{N-1} & x_{N-2} & \cdots & x_0. \\
\end{array}
\]

c) \((M + 1)\)-positional fractional number

\[
\begin{array}{cccc}
\text{Sign field} & \text{Numeric field} \\
& x_S. & x_1 & x_2 & \cdots & x_M \\
\end{array}
\]
Floating point format of a number

- Normalised floating point representation
  \[ \text{radix}^{-1} \leq \text{mantissa} < 1 \]
Fixed-point arithmetic operations

• Addition of binary numbers with sign in all notations
• Addition of binary coded decimal numbers with sign
• Multiplication of binary numbers with sign
• Division of binary numbers with sign
• Multiplication and division of binary coded decimal numbers with sign
Fixed-point addition

• Operations on numbers without sign can be treated as operations on numbers with sign
• Subtraction can be treated as addition with opposite sign
• Beware of overflow!
  – When added numbers have the same sign the magnitude of the result can exceed the maximal number that can be represented for the selected number of bits and occurring overflow can cause wrong result
  – Solutions
    • If possible, one extra digit should be added as the most significant, equal either 0 or \((R-1)\), depending on sign and notation
    • Using overflow indicator or modifiable notation
Fixed-point binary addition

• 1’s complement notation (DRC)
  – operation on all bits, including the sign bit
  – if a carry occurs from the sign position, it is added at the least significant position to obtain final result
  – when numbers with the same signs are added and the result has the opposite sign it indicates the occurrence of overflow and wrong result
Example for DRC notation

- For \((X)_2 = 1101.11011\) and \((Y)_2 = 1000.01101\)

\[
\begin{align*}
X + Y &= \text{001101.11011} + \text{001000.01101} \\
&= \text{010110.01000} \\
-X - Y &= (-X) + (-Y) \\
(-X)_2^{DRC} &= \text{110010.00100} \\
(-Y)_2^{DRC} &= \text{110111.10010} \\
&= \text{101001.10111}
\end{align*}
\]

\[
\begin{align*}
X - Y &= X + (-Y) \\
(X)_2^{DRC} &= \text{001101.11011} \\
(-Y)_2^{DRC} &= \text{110111.10010} \\
&= \text{000101.01110} \\
&= \text{001000.01101}
\end{align*}
\]

\[
\begin{align*}
- X + Y &= (-X) + Y \\
(-X)_2^{DRC} &= \text{110010.00100} \\
(Y)_2^{DRC} &= \text{001000.01101} \\
&= \text{111010.10001}
\end{align*}
\]
Fixed-point binary addition

• 2’s complement notation (RC)
  – operation on all bits, including the sign bit
  – if a carry occurs from the sign position, it should be disregarded
  – when numbers with the same signs are added and the result has the opposite sign it indicates the occurrence of overflow and wrong result
Example for RC notation

- For \((X)_2 = 1101.11011\) and \((Y)_2 = 1000.01101\)

\[
\begin{align*}
X + Y \quad & \quad (X)_2^{RC} = \begin{cases} 001101.11011 \\ 001000.01101 \end{cases} + \begin{cases} 010110.01000 \\ 010111.10011 \end{cases} \\
\end{align*}
\]

\[
\begin{align*}
X - Y = X + (-Y) \quad & \quad \begin{cases} 001101.11011 \\ 110111.10011 \end{cases} + \begin{cases} 000101.01110 \end{cases} \\
\end{align*}
\]

\[
\begin{align*}
-X - Y = (-X) + (-Y) \quad & \quad \begin{cases} 110010.00101 \\ 110111.10011 \end{cases} + \begin{cases} 101001.11000 \end{cases} \\
-X + Y = (-X) + Y \quad & \quad \begin{cases} 110010.00101 \\ 001000.01101 \end{cases} + \begin{cases} 111010.10010 \end{cases} \\
\end{align*}
\]
Fixed-point binary addition

• Sign-Magnitude (SM) notation
  – operation in separate, yet dependent parts for
    • sign bits and
    • magnitude
  – algorithm requires
    • comparison of bits
    • calculation of complements
    • detection of carry
Addition in SM notation

• 1st step is to compare sign bits
  – if sign bits are equal
    • add magnitudes to obtain the magnitude of the result
    • the sign is the same as for input numbers
  – if sign bits differ
    • to the magnitude of augend we add the \((R-1)’s\) complement of the addend
    • when carry occurs from this addition
      – the obtained number is the magnitude of the output result
      – the sign of the output result is the same as augend’s
    • when there is no carry from the addition
      – the magnitude of the output result equals to \((R-1)’s\) complement of the obtained number
      – the sign of the output result is the same as addend’s
Example for SM notation

• For \((X)_2 = 1101.11011\) and \((Y)_2 = 1000.01101\),
\[X + Y = (\overline{X}) + (\overline{Y})\]
\[
(X)_{SM}^2 = 001101.11011 \\
(Y)_{SM}^2 = 001000.01101
\]
\[z_S = x_S = y_S\]
\[
|X| = 01101.11011 \\
|Y| = 01000.01101
\]
\[|Z| = 10110.01000
\]
\[z_S = 0 \\
(Z)_{SM}^2 = 010110.01000
\]
\[z_S = 1 \\
(Z)_{SM}^2 = 110110.01000\]
Example for SM notation

- For \((X)_2 = 1101.11011\) and \((Y)_2 = 1000.01101\)

\[
X - Y = X + (-Y) \quad -X + Y = (-X) + Y
\]

\[
(X)_{2}^{SM} = 001101.11011 \quad (-X)_{2}^{SM} = 101101.11011
\]

\[
(-Y)_{2}^{SM} = 101000.01101 \quad (Y)_{2}^{SM} = 001000.01101
\]

\[z_S = x_S = 0 \quad (Z)_{2}^{SM} = 000101.01110\]

\[z_S = x_S = 1 \quad (Z)_{2}^{SM} = 100101.01110\]
Example for SM notation

- For \((X)_2 = 1101.11011\) and \((Y)_2 = 1000.01101\)

\[-Y + X = (-Y) + X\]

\((-Y)^{SM}_2 = 101000.01101\)
\((X)^{SM}_2 = 001101.11011\)

\(y_S \neq x_S\)

\[
\begin{align*}
|Y| &= 01000.01101 \\
|X| &= 10010.00100 \\
|Z| &= 11010.10001
\end{align*}
\]

\(\neg x_S = x_S = 0\)
\(|Z| = 00101.01110\)
\((Z)^{SM}_2 = 000101.01110\)